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Short Papers

An Expansion for the Fringing Capacitance

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Abstract—**The first twelve terms in an expansion of the "approximate fringing capacitance" in powers of** $\exp(-\pi s/b)$ **are given explicitly as functions of** t/b. **Comparison with exact values shows agreement within 0.06 percent for** $s/b > 0.2$ **and** $t/b < 0.5$. **In the extreme case considered,** $s/b = 0.1$ **and** $t/b = 0.5$, **the error is less than 2.3 percent.**

Introduction

The "approximate fringing capacitance" C'_{f_0}, as defined by Cohn [1] and Getsinger [2] is useful in a number of ways in the approximation of the capacitance of certain rectangular coaxial structures. Explicit formulas for it have been given by Cockroft [3], Getsinger [2], and Riblet [4]. These formulas express C'_{f_0} in terms of two independent real parameters a and k. The normalized geometric parameters, t/b and s/b of Fig. 1 are also given in terms of these parameters, but, before C'_{f_0} can be found for a given geometry, these equations must be inverted in some way and a and k determined for the given values of t/b and s/b.

Heretofore this determination has required some form of graphical or numerical trial and error process. Recently, Riblet [5], however, has shown how for large values of s/b, these equations can be inverted. In this note these values for a and k are substituted directly in the formula for C'_{f_0} and an expansion obtained for C'_{f_0} in powers of $\exp(-\pi s/b)$, whose coefficients are given functions of t/b, which has useful accuracy for s/b as small as 0.1.

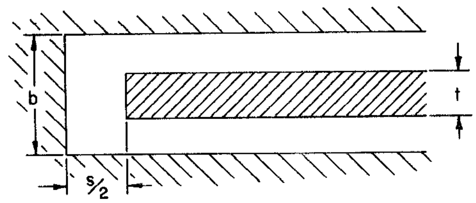

Fig. 1. Fringing capacitance cross section.

The Problem

It is not difficult, following Bowman [6] to express the quantities b, s, and t, of Fig. 1, except for a scale factor, in terms of two independent real parameters a and k, where k is the modulus of the Jacobi elliptic functions involved. It is no restriction to assume that $0 \leqslant k \leqslant 1$ and $0 \leqslant a \leqslant K$. Then

$$b = 2K\left\{\frac{\text{sn}a\,\text{dn}a}{\text{cn}a} - Z(a)\right\} - \frac{\pi a}{K} + \pi \tag{1}$$

$$s = 2K\left\{\frac{\text{sn}a\,\text{dn}a}{\text{cn}a} - Z(a)\right\} \tag{2}$$

$$t = 2K'\left\{\frac{\text{sn}a\,\text{dn}a}{\text{cn}a} - Z(a)\right\} - \frac{\pi a}{K}. \tag{3}$$

The approximate odd-mode fringing capacitance, C'_{f_0} for this geometry is given in terms of the same parameters a and k by

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the expression

$$C'_{f_0}=\frac{2}{\pi}\left\{(K-a)\left(\frac{\operatorname{sn}a\operatorname{dn}a}{\operatorname{cn}a}-Z(a)\right)-\log(k\operatorname{sn}a\operatorname{cn}a)-2\log(\theta_n(a))\right\}. \quad (4)$$

Here the functions are all those which are familiar from Jacobi's theory of elliptic functions and $\theta_n(a)=\theta(a)/\theta(0)$. Now since we have $\operatorname{sn}a=H(a)/\sqrt{k}\,\theta(a)$ and $\operatorname{cn}a=\sqrt{k'}\,H_1(a)/\sqrt{k}\,\theta(a)$ by Hancock [7, p. 244] and $\theta(0)=\sqrt{2k'K/\pi}$ by Cayley [8, p. 149] (4) may be written

$$C'_{f_0}=\frac{2}{\pi}\left\{(K-a)\left(\frac{\operatorname{sn}a\operatorname{dn}a}{\operatorname{cn}a}-Z(a)\right)-\log\left(\frac{\pi}{2K\sqrt{k'}}H(a)H_1(a)\right)\right\}. \quad (5)$$

Now it is clear from (1) and (3) that

$$\frac{\operatorname{sn}a\operatorname{dn}a}{\operatorname{cn}a}-Z(a)=\frac{\pi}{2KK'}\left\{\frac{(K-a)t+ab}{b-t}\right\}=\frac{\pi}{K'}\frac{\beta-1}{2}+\frac{\pi a}{2KK'} \quad (6)$$

if $\beta=1/(1-t/b)$. So that

$$C'_{f_0}=\frac{2}{\pi}\left\{\frac{\pi(K-a)}{K'}\frac{\beta-1}{2}+\frac{\pi a}{2K'}-\frac{\pi a^2}{2KK'}-\log\left(\frac{\pi}{2K\sqrt{k'}}H(a)H_1(a)\right)\right\}. \quad (7)$$

Thus C'_{f_0} has been expressed entirely in terms of the Jacobian Theta functions, $H(a)$ and $H_1(a)$ and the parameters a, k, and t/b. Then introducing $p=\exp(-\pi(K-a)/K')$ as defined in [5] and substituting

$$H(a)H_1(a)=\frac{K}{K'}\exp\left(-\frac{\pi a^2}{2KK}\right)\frac{H(ja,k')\theta(ja,k')}{j} \quad (8)$$

as given by [7, p. 296], equation (7) becomes

$$C'_{f_0}=\frac{2}{\pi}\left\{-\frac{\beta-1}{2}\log p-\log\left(\exp\left(-\frac{\pi a}{2K'}\right)\frac{\pi H(ja,k')\theta(ja,k')}{2K'\sqrt{k'}j}\right)\right\}. \quad (9)$$

Now from [7, p. 238],

$$H(ja,k')=-\frac{2q'^{1/4}}{j}(\sinh\nu'-q'^2\sinh 3\nu'+q'^6\sinh 5\nu'-q'^{12}\cdot\sinh 7\nu'+q'^{20}\sinh 9\nu'+\cdots)$$

$$\theta(ja,k')=1-2q'\cosh 2\nu'+2q'^4\cosh 4\nu'-2q'^9\cdot\cosh 6\nu'+2q'^{16}\cosh 8\nu'+\cdots \quad (10)$$

where $\nu'=\pi a/2K'$. Then substituting, the form of C'_{f_0} useful for finding the desired expansion is

$$C'_{f_0}=\frac{2}{\pi}\left\{-\frac{\beta-1}{2}\log p-\log\frac{\pi q'^{1/4}}{K'\sqrt{k'}}-\log e^{-(\pi a/2K')}(\sinh\nu'-q'^2\sinh 3\nu'+\cdots)-\log(1-2q'\cosh 2\nu'+2q'^4\cosh 4\nu'+\cdots)\right\}. \quad (11)$$

How an expansion for p in powers of q' can be obtained was the principal result of [5]. The expansion for $\pi q'^{1/4}/K'\sqrt{k'}$ in powers of q' can be determined from [7, pp. 241, 400]. The last two terms of (11) can be expressed as power series in q' whose coefficients contain positive and negative powers of p. Thus these terms as well can be expressed as powers series in q' whose coefficients are functions of t/b. Having found then a power series in q' for C'_{f_0} it only remains to return to [5] where it was shown how q' can be expressed as an odd power series in $\exp(-\pi s/b)$. When this series for q' is substituted in the series already found for C'_{f_0} in terms of q', the desired result is obtained.

The Result

Determining the first four terms of the desired expansion by explicit derivation was carried through without any special difficulty and revealed the general form of the coefficients. These terms, however, become increasingly involved and the results might not warrant the manual labor that would be required. Consequently the majority of the terms were obtained by curve fitting with the help of a digital computer.

It was found that

$$C'_{f_0}\approx\frac{2}{\pi}\sum_{i=0}^{11}\frac{A_i}{D_i}\exp\left(-\frac{i\pi s}{b}\right)$$

$$D=(\beta+1)^{1+1/\beta}(\beta-1)^{1-1/\beta}$$

$$A_0=\frac{\beta+1}{2}\log(\beta+1)-\frac{\beta-1}{2}\log(\beta-1)$$

$$A_1=4\beta^2$$

$$A_2=4\beta^2(\beta^2+1)$$

$$A_3=\frac{16}{3}\beta^2(\beta^4+3)$$

$$A_4=4\beta^2(\beta^6+3\beta^4-\beta^2+13)$$

$$A_5=\frac{8}{5}\beta^2(3\beta^8+70\beta^4-120\beta^2+175)$$

$$A_6=\frac{16}{3}\beta^2(\beta^{10}+3\beta^8-6\beta^6+102\beta^4-203\beta^2+231)$$

$$A_7=\frac{32}{21}\beta^2(3\beta^{12}+231\beta^8-1176\beta^6+4305\beta^4-6664\beta^2+4837)$$

$$A_8=\frac{4}{3}\beta^2(3\beta^{14}+21\beta^{12}-69\beta^{10}+1989\beta^8-10\,479\beta^6+31\,847\beta^4-45\,135\beta^2+27\,967)$$

$$A_9=\frac{4}{9}\beta^2(13\beta^{16}+1764\beta^{12}-18\,288\beta^{10}+125\,766\beta^8-458\,592\beta^6+977\,940\beta^4-1\,092\,528\beta^2+529\,461)$$

$$A_{10}=\frac{8}{15}\beta^2(9\beta^{18}+45\beta^{16}-300\beta^{14}+16\,500\beta^{12}-163\,170\beta^{10}+928\,10\beta^8-2\,981\,020\beta^6+5\,594\,852\beta^4-5\,617\,375\beta^2+2\,418\,965) \quad (12)$$

$$A_{11}=\frac{16}{165}\beta^2(45\beta^{20}+15\,675\beta^{16}-261\,360\beta^{14}+2\,937\,330\beta^{12}-1\,8857\,520\beta^{10}+75\,379\,590\beta^8-187\,182\,160\beta^6+283\,129\,297\beta^4-237\,922\,960\beta^2+86\,694\,223).$$

Table I gives the exact values of C'_{f_0} for a range of values of $t/b<0.5$, for $s/b=0.1$, 0.15, and 0.2 together with approximate values obtained from (12). The upper figure in each group is exact, the next figure is given by (12) while the bottom figure neglects A_{11}. It is clear that C'_{f_0} is given with increasing accuracy by (12) as s/b increases and t/b decreases. Even in the extreme

TABLE I
EXACT AND APPROXIMATE C'_{f0}

t/b	s/b		
	.1	15	2
.1	2.26132	1.72183	1.42231
	2.23063	1.71846	1.42189
25	3 66637	2.63689	2 09720
	3.59531	2.62912	2.09625
.5	5 97022	4.13495	3.20397
	5.83221	4.11990	3 20213
	5.78035	4 11069	3.20049

case, where $s/b=0.1$ and $t/b=0.5$, the accuracy is comparable to that achievable with Getsinger's charts [2], and more than adequate for most engineering purposes.

The coefficients of β^i are all given in integral form and are believed to be those which one would have obtained had they been derived by step-by-step algebraic substitution. In the programs used an error of only one in the nine place integers is easily detected. It was reassuring to find, however, that after the coefficients had been determined, that terms by term (12) approaches the known limit,

$$-\frac{2}{\pi}\log\frac{(1-\exp(-\pi s/b))}{2}$$

as $t/b \to 0$, although this value of t/b was not used in the curve fitting process by which they were found.

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Diathermy Applicators with Circular Aperture and Corrugated Flange

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***Abstract*—A design method and experimental results for a direct-contact circular aperture applicator are provided. The aperture is excited in the TE_{11} mode; a corrugated flange surrounding the aperture improves the uniformity of the heating pattern and limits leakage. The performance of the applicators operating in the *S* band (2.45 GHz) and the *X* band (9.96 GHz) has been tested using a short monopole probe and a thermographic camera. The heating patterns obtained by the two methods are in agreement within the experimental errors. The applicators are suitable for clinical use, as they are lightweight and rugged, and capable of delivering a desired energy dose effectively, thanks to a relatively small standing-wave ratio (SWR <2) and very low leakage.**

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I. INTRODUCTION

Numerous microwave diathermy applicators have recently been developed [1]–[6]. This has been prompted by some indication that local hyperthermia induced by microwave energy may be used in the treatment of certain malignant tumors. Also, the requirements of the proposed U.S. draft standard for microwave diathermy equipment [7] and the proposed Canadian regulations for these devices [8] require new design approaches as follows: 1) effective heating in the plane parallel to the applicator aperture; 2) deposition of microwave energy at a required rate in muscle or tumor with minimal heating of fat and skin or tumor surrounding tissues; and 3) minimum leakage. The first objective can be met by selecting appropriate aperture size and operating frequency. The penetration depth and the resulting "in-depth heating" of muscle are primarily determined by the frequency of operation [2], [3], [9]–[11]. However, the relative dimensions of the radiating aperture (as compared to the wavelength) also play a significant role in the determination of the in-depth heating profile. For instance, for a rectangular aperture it was found that the optimum profile (i.e., providing the minimum ratio of fat-to-muscle heating, and deep muscle heating) is obtained when the aperture height is between one and two wavelengths and the width about one wavelength in fat, assuming a TE_{11}-mode distribution on the aperture [2], [11].

The heating profile in the plane parallel to the aperture is determined by the dimensions and the electric-field distribution on the aperture. The immediate surroundings of the aperture (flange) also affect the profile. To obtain symmetrical heating, cross polarization and circular polarization have been employed [1], as well as dielectric loading [12], [13] and multimode operation [4].

In this paper the design and experimental results for a corrugated flange applicator are presented. Two methods, namely a thermographic camera technique [1], [10] and a short *E*-field probe method [14] were used to determine the heating patterns of the applicators operating in the *S* band (2.45 GHz) and the *X* band (9.6–10 GHz). A comparison of the two methods is given.

II. APPLICATOR DESIGN

Two applicators, one operating at a frequency of 2.45 GHz and one operating in a frequency range 9.6–10 GHz have been designed. The design procedures described here can be in principle used also at lower microwave frequencies where deeper penetration of microwave energy is possible. An applicator operating at 915 MHz is presently being investigated. External views of the applicators are shown in Fig. 1. A circular aperture, and the fundamental mode (TE_{11}) in the waveguide and on the aperture were employed. The waveguide is terminated by the corrugated flange consisting of four grooves.

Selection of the diameter of the waveguide and the radiating aperture (D) is based on the following considerations. The frequency of operation must be above the cutoff frequency of the fundamental mode, and preferably it should be below the cutoff of the next higher order mode (TM_{01}). For antenna beam

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